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On orientifolds of WZW models and their relation to geometry

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ABSTRACT: We investigate D-branes in orientifolds of WZW models. A connection between the conformal field theory approach to orientifolds and the target space motivated analysis is established. In particular, we associate previously constructed crosscap states to involutions of the group manifold and their fixed point sets. Whereas our analysis of D-branes in orientifolds of general WZW models is restricted to special D0-branes, we investigate all symmetry preserving branes of $SU(2)$ -orientifolds in detail. For that case, the location of the orientifold fixed point set is independently determined by scattering localized graviton wave packets.

KEYWORDS: D-branes, Conformal Field Models in String Theory.

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1. Introduction

In the last two years considerable progress has been achieved in the understanding of D-branes on group manifolds [1]–[7], in particular on the geometrical interpretation of conformal field theory results. In this paper, we start a similar program for orientifolds of group manifolds.

In conformal field theory, D-branes are characterized by boundary conditions for the elements of the chiral algebra. For maximally symmetric gluing conditions $J(z) = \bar{J}(z)$ on the upper half plane an exact conformal field theory description in terms of boundary states had been given starting with the work of Cardy. More recently, it was shown how the geometry of the associated D-branes is encoded in the gluing conditions. In particular, the gluing conditions constrain the end points of open strings to conjugacy classes of the group manifold [1].

In the case of orientifolds, the underlying theory is modded out by an involution involving world sheet parity and possible additional actions on the target manifold. The target space action can in general have fixed points. In flat space, we associate to the fixed points of such involutions “orientifold planes” in the target geometry. In this way, orientifolds have a geometrical meaning in terms of target data.

In conformal field theory, orientifolds are described by crosscap states. The construction of crosscaps on group manifolds and in general rational conformal field theory has been investigated in [8]–[14]. In this paper we shall match the conformal field theory data of the orientifold encoded in a choice of consistent crosscap state to the geometric description in terms of a target space involution.

The first part of the paper is devoted to the case of an arbitrary WZW model. We discuss suitable target involutions that preserve all symmetries, and review some essential points of the work of [8]–[13], [15]–[18] on crosscap states. We then map crosscap states to suitable target involutions, thus connecting the world sheet and target space motivated analysis. The tool is the interpretation of special Möbius strip amplitudes.

In the second part, we investigate orientifolds of $SU(2)$ in greater detail. For $SU(2)$ the conjugacy classes are generically two-spheres in $S^3 \sim SU(2)$ and two additional conjugacy classes are points. These conjugacy classes can be wrapped by symmetry preserving D-branes. For $SU(2)$ there are two different target involutions which can be divided out while preserving the $SU(2)$ current algebra. Both of these have fixed point sets, to which orientifold loci (which are not planes) can be associated.

The reflections of S^3 act naturally on the open string sector of the theory. Geometrically, one can determine how conjugacy classes transform under the involution. If the conjugacy class is mapped to itself, the reflection induces an involution on the algebra of functions on the conjugacy class, which is the algebra of spherical harmonics. From a conformal field theory point of view, this induced involution corresponds to the action of Ω in the open string sector with boundary conditions determined by the conjugacy class. We compare the geometrically induced involution to the action of Ω as determined by the crosscap and boundary state for all boundary conditions.

The location of the orientifold can be determined independently by scattering localized graviton wave packets from it. We perform this computation in the case of $SU(2)$, confirming the results of the earlier sections.

2. General considerations

2.1 Fixed point sets and gluing conditions

The symmetries of a WZW model consist of a left- and a right-moving current algebra. In terms of group elements, the currents can be written as

$$J = k g^{-1} \partial g, \quad \bar{J} = -k \bar{\partial} g g^{-1}. \quad (2.1)$$

For a symmetry preserving orientifold induced by an involution Ω , the left and right moving current algebras must be interchanged under the orientifold action. In the flat space case one is used to thinking of orientifolds in terms of orientifold planes in the target geometry. The aim of the following sections is to find a similar geometrical interpretation in terms of the target space data also when the target space is a group manifold. Stated differently, one requires an involution on the target, such that the induced action on the currents is to exchange the left and right moving current algebra. The fixed point set is then interpreted geometrically as the location of the orientifold.

An example for such an involution is [18, 19, 20]:

$$\omega_+ : g(z, \bar{z}) \rightarrow g^{-1}(\bar{z}, z). \quad (2.2)$$

This involution exists generically for every WZW model. It maps conjugacy classes to

conjugacy classes. The precise form of the fixed point set depends on the particular group under consideration. The identity $\mathbb{1}$ of the group is always part of the fixed point set, but the involution might leave other conjugacy classes fixed.

In addition, there can exist other involutions which can be divided out. If the group manifold has a non-trivial center \mathbb{Z} , then the following involution can also be considered:

$$\omega_z : g(z, \bar{z}) \rightarrow zg^{-1}(\bar{z}, z) \quad z \in \mathbb{Z}. \quad (2.3)$$

In order to map J and \bar{J} on each other (the case we want to consider in this paper), z has to square to the identity, $z^2 = \mathbb{1}$. The fixed point set consists of all group elements squaring to z . In general, it can have several disconnected components consisting of various grassmannians.

2.2 Crosscap states for WZW models

Starting from a closed string vacuum whose closed string spectrum is encoded in the charge conjugation modular invariant, open descendants containing unoriented strings have been constructed in [8]–[13], and [17, 15]. These provide candidates for a conformal field theory description of orientifolds, to which we later want to associate the involutions discussed above. We will therefore briefly review the relevant formulas from those papers. The construction of orientifolds involves unorientable worldsheets. The basic ingredient is the crosscap \mathbb{RP}_2 , the simplest non-orientable surface. The introduction of the crosscap breaks the $\mathcal{A} \times \tilde{\mathcal{A}}$ symmetry of the model, where \mathcal{A} ($\tilde{\mathcal{A}}$) denote the left (right) moving current algebra. Here, we are considering crosscaps that leave one copy of the current algebra unbroken. The condition for this is that

$$(J_n - (-1)^n \bar{J}_{-n}) |C, j\rangle = 0. \quad (2.4)$$

The $|C, j\rangle$ are crosscap Ishibashi states normalized such that

$$\langle C, j | e^{-\frac{i}{2}(L_0 + \bar{L}_0)} |C, l\rangle = \delta_{j,l} \chi_j(i\tilde{t}). \quad (2.5)$$

Uncapitalized letters j, l, \dots label the representations of the chiral algebra, in the case at hand the irreducible representations of the current algebra \mathcal{A} . Crosscap states are then linear combinations of the crosscap Ishibashi states leading to consistent Klein bottle and Möbius strip amplitudes. A formula for consistent crosscaps in the context of RCFT has been given by [17, 10]

$$|C_{gen}\rangle = \sum_j \frac{P_{j0}}{\sqrt{S_{j0}}} |C, j\rangle, \quad (2.6)$$

where

$$P = T^{1/2} S T^2 S T^{1/2}. \quad (2.7)$$

The transverse channel of the Klein bottle describes the propagation of closed strings on a tube terminating on two crosscaps. This amplitude follows immediately from (2.5)

and (2.6). Using an S modular transformation, one obtains the direct channel of the Klein bottle:

$$K = \sum_i Y_{00}^i \chi_i, \quad (2.8)$$

where Y_{00}^i is the Frobenius Schur indicator [12, 21] introduced for RCFT in [22]. It is 1 when i is a real representation, -1 if i is pseudo-real and 0 if it is complex [22]. Y_{00}^i is a special component of the following integer valued [10, 23] tensor:

$$Y_{ij}^k = \sum_m \frac{S_{im} P_{jm} P_{km}^\dagger}{S_{0m}}. \quad (2.9)$$

This crosscap is consistent with all Cardy boundary states. The amplitudes corresponding to the Möbius strip are in the tree channel given by cylinders capped off by a crosscap at one end and by a boundary state at the other end. The transformation of these amplitudes to the open string sector results in a one-loop amplitude with an Ω insertion in the open string sector. The matrix mediating between the two channels is the matrix P of equation (2.7).

We have seen that in the case that the group has a non-trivial center, in particular if the center contains a \mathbb{Z}_2 subgroup, there are additional possibilities for orientifolds. In conformal field theory, the existence of a non-trivial abelian center, for example \mathbb{Z}_N for $SU(N)$, manifests itself in the existence of simple currents. These are special primaries whose fusion with any other field yields exactly one field. The abelian group generated by these currents is isomorphic to the center of the group. From this point of view, it has to be expected that simple currents enter the discussion of orientifolds. Indeed, additional consistent crosscap states in models with non-trivial simple current groups were discovered in [12, 9]. The explicit expression for a crosscap involving a simple current L is [12]

$$|C_L\rangle = \sum_j \frac{P_{jL}}{\sqrt{S_{j0}}} |C, j\rangle, \quad (2.10)$$

leading to the modified Klein bottle

$$K_L = \sum_i Y_{LL}^i \chi_i = \sum_i e^{2\pi i Q_L(i)} Y_{00}^i \chi_i. \quad (2.11)$$

Here, $Q_L(i)$ is the monodromy charge of the field i with respect to the current L :

$$Q_L(i) = h_L + h_i - h_{L \times i} \mod 1. \quad (2.12)$$

2.3 Matching Crosscap states and fixed point sets

A priori, it seems suggestive that the crosscap $|C_{gen}\rangle$ in equation (2.6), which exists in any WZW model should provide the world sheet description of the target involution $g \rightarrow g^{-1}$, which exists for all group manifolds. The crosscap $|C_L\rangle$ should correspond to the involution $g \rightarrow zg^{-1}$, as can be expected from the connection between simple current groups and non-trivial centers of groups.

This can be confirmed by the following argument: Consider the brane located at the identity. For any group manifold, the identity represents a conjugacy class consisting of

one point. The D-brane sitting at that point is described by the Cardy state labeled by the vacuum character

$$|0\rangle = \sum_i \sqrt{S_{0i}} |i\rangle, \quad (2.13)$$

where $|i\rangle$ is the Ishibashi state built on the primary i . The open string channel of the cylinder with those boundary conditions contains only the character χ_0 .

The involution $g \rightarrow g^{-1}$ leaves the identity fixed, and therefore, the image of the brane $|0\rangle$ is $|0\rangle$. The Möbius amplitude can be interpreted as an amplitude between the brane and its image. In the present situation, it can be concluded that the character χ_0 is also the only one appearing in the Möbius amplitude. Using the matrix P , one obtains the transverse amplitude

$$M_0 = \chi_0 \longleftrightarrow \sum_j P_{0j} \chi_j = \sum_j \sqrt{S_{0j}} \Gamma_j \chi_j. \quad (2.14)$$

In the last step, the amplitude in the transverse channel was expressed as an overlap of the boundary state $|0\rangle$ and a crosscap $|C\rangle = \sum \Gamma_j |C, j\rangle$. It can be read off that the coefficient of the crosscap $|C\rangle$ are given by

$$\Gamma_j = \frac{P_{0j}}{\sqrt{S_{0j}}}. \quad (2.15)$$

This supports that the crosscap $|C_{gen}\rangle$ given in the previous section corresponds to the fixed point set of $g \rightarrow g^{-1}$. Of course, there is at this point of the discussion a choice of an overall sign, and also the crosscap $-|C_{gen}\rangle$ would be located at the same fixed point set. The sign reverses the projection on Ω -invariant states in the open string sector and would be determined in a full string theory by tadpole cancellation conditions.

The involution $g \rightarrow zg^{-1}$ maps the identity to the group element z . The D-brane located at z is represented by the Cardy state which carries the representation label of the simple current L corresponding to z :

$$|L\rangle = \sum_j e^{2\pi i Q_L(j)} \sqrt{S_{0j}} |j\rangle. \quad (2.16)$$

To compare to Cardy's standard formula, note that $S_{Lj} = \exp(2\pi i Q_L(j)) S_{0j}$ [24]. The open string spectrum with boundary conditions $|0\rangle$ and its image $|L\rangle$ consists of the single character χ_L . As before, it can be concluded that the only character appearing in the one loop channel of the Möbius amplitude is χ_L

$$M_0^L = \chi_L \longleftrightarrow \sum_j P_{Lj} \chi_j = \sum_j \sqrt{S_{0j}} \Gamma_j \chi_j. \quad (2.17)$$

The factorization in the transverse channel gives

$$\Gamma_j = \frac{P_{Lj}}{\sqrt{S_{0j}}}. \quad (2.18)$$

For consistency, since z and 1 are images of *each other*, the Möbius strip with boundary condition $|L\rangle$ equals the one with boundary condition $|0\rangle$ and accordingly it must be pos-

sible to factorize the amplitude in the transverse channel also into the crosscap state and the boundary state $|L\rangle$. This is indeed possible, since $P_{Lj} = 0$ whenever $Q_L(j) \in \mathbb{Z} + 1/2$ (and L is (half-) integer spin).

The conclusion we can reach is therefore that the involution $g \rightarrow zg^{-1}$ corresponds to the crosscap state $|C_L\rangle$, whereas $|C_{genrangle}$ corresponds to $g \rightarrow g^{-1}$.

Turning the chain of arguments around, the above discussion could lead to an alternative derivation of the coefficients of the crosscap state: One starts with an involution on the group manifold and determines the spectrum of open string states and the action of Ω for a particular boundary condition. Here, we have chosen the boundary condition $|0\rangle$. One then transforms to the transverse channel and factorizes the result into a boundary and a crosscap state. The coefficients for the crosscap state can be read off from the amplitude. In principle, one can perform this type of analysis for any boundary condition, and the analysis of those leads to important consistency checks. In the case of $SU(2)$, which we consider in the next section, we will in fact study all boundary conditions, leading to a detailed match of all amplitudes with geometrical expectations.

3. Example: $SU(2)$

3.1 Involutions and crosscaps

Let us apply these considerations to the case of $SU(2)_k$, where throughout the rest of this paper it is assumed that k is even. See [18, 20, 19] for earlier discussions of $SU(2)$ orientifolds in the context of five-branes and [14] for an early discussion of crosscap states under consideration in this section.

The center of $SU(2)$ consists of the two elements $\{\mathbb{1}, -\mathbb{1}\}$. Accordingly, there are two possible involutions (2.3) which can be used for orientifolds. The fixed point set of the inversion $g \rightarrow g^{-1}$ consists of $\pm\mathbb{1}$, whereas the fixed point set of $g \rightarrow -g^{-1}$ consists of the conjugacy class of the group element

$$g_{eq} = \begin{pmatrix} e^{\frac{\pi i}{2}} & 0 \\ 0 & e^{-\frac{\pi i}{2}} \end{pmatrix}, \quad (3.1)$$

which is the conjugacy class located at the equator of S^3 .

Turning to the CFT point of view, recall that the characters of $SU(2)_k$ are labeled by $j = 0, 1/2, 1, \dots, k/2$. Accordingly, there are $k+1$ Cardy states $|J\rangle$ wrapping the $k+1$ integral conjugacy classes that exist at finite k . More precisely, the Cardy state $|J\rangle$ wraps the conjugacy class represented by

$$g_J = \begin{pmatrix} e^{\frac{2\pi i J}{k}} & 0 \\ 0 & e^{-\frac{2\pi i J}{k}} \end{pmatrix}. \quad (3.2)$$

The non-vanishing entries of the matrix P for $SU(2)_k$ (k even) are given by

$$P_{jl} = \frac{2}{\sqrt{k+2}} \sin \frac{\pi(2j+1)(2l+1)}{2(k+2)} \quad \text{for } j+l \in \mathbb{Z}. \quad (3.3)$$

There is a simple current in the theory which carries the representation $k/2$ and maps the primaries j to $k/2 - j$. The simple current group $\{0, k/2\}$ is therefore \mathbb{Z}_2 and isomorphic

to the center of the group. According to the above discussion, the crosscap state located at the equator of the group manifold is

$$|C_{eq}\rangle = \left(\frac{2}{k+2}\right)^{1/4} \sum_{j=0}^{k/2} (-1)^j \cot^{1/2} \left(\frac{(2j+1)\pi}{2(k+2)} \right) |C, j\rangle, \quad (3.4)$$

and the one located at ± 1 is

$$|C_{\pm}\rangle = \left(\frac{2}{k+2}\right)^{1/4} \sum_{j=0}^{k/2} \tan^{1/2} \left(\frac{(2j+1)\pi}{2(k+2)} \right) |C, j\rangle. \quad (3.5)$$

3.2 Geometry and Möbius strips

So far, the discussion of Möbius strips has been restricted to the evaluation and interpretation of amplitudes with the boundary condition $|0\rangle$. In the $SU(2)$ case we are now going to interpret the Möbius strips with all boundary conditions. This discussion is somewhat similar to that in [25] for \mathbb{Z}_2 orbifolds of $SU(2)$.

Recall some basic facts about the geometry of the conjugacy classes [2, 3]. There are two conjugacy classes, ± 1 , which are points, and $k-1$ conjugacy classes which are spheres $S^2 \subset S^3$. Since we already evaluated the Möbius strips for the point-like branes we turn to the conjugacy classes which are spheres. According to the analysis of [2, 3] these spheres are “fuzzy” which amounts on the level of the algebra of functions to a truncation. More precisely, the algebra of functions of a sphere is spanned by spherical harmonics $Y^{j,m}$, where j, m are integers and $|m| \leq j$. For the fuzzy sphere describing the geometry of the conjugacy class represented by the group element g_J on (3.2) the label j takes integer values up to a maximum value of $2J$.

In the case that the conjugacy class J is either point-wise or set-wise fixed under an involution, the fuzzy sphere algebra inherits an involution from that reflection. In our situation, the fuzzy sphere located at the equator is the only sphere which is fixed under both involutions. More precisely, it is point-wise fixed under $g \rightarrow -g^{-1}$ and set-wise fixed under $g \rightarrow g^{-1}$. Under the latter involution, individual group elements of the conjugacy class get mapped as $g \rightarrow -g$. The inherited action on the algebra of spherical harmonics is

$$\begin{aligned} Y^{jm} &\rightarrow Y^{jm} \quad \text{for } g \rightarrow -g^{-1} \\ Y^{jm} &\rightarrow (-1)^j Y^{jm} \quad \text{for } g \rightarrow g^{-1}. \end{aligned} \quad (3.6)$$

Turning to the conformal field theory point of view, the action of the inherited involution on the spherical harmonics should be compared to the action of Ω on the open string sector primaries with appropriate boundary conditions. This action can be read off from the open string channel of the Möbius strip with boundary conditions $|k/4\rangle$, where $|k/4\rangle$ is the Cardy state representing the D-brane wrapping the equator:

$$\left| \frac{k}{4} \right\rangle = \left(\frac{2}{k+2}\right)^{1/4} \sum_{j=0}^{k/2} (-1)^j \left(\sin \frac{\pi(2j+1)}{k+2} \right)^{-1/2} |j\rangle. \quad (3.7)$$

Note that this Cardy state exists only for k even. The cylinder amplitude for this boundary state contains all integer representation labels,

$$Z_{cyl}^{k/4} = \sum_{j=0}^{k/2} \chi_j \left(\frac{it}{2} \right). \quad (3.8)$$

According to the above geometric consideration, the Möbius strip should also contain all integer representation labels, but this time with an insertion of Ω , as determined by equations (3.6). Indeed one can confirm that

$$M_{k/4}^L = \sum_{j=0}^{k/2} \chi_j \left(\frac{1+it}{2} \right) \longleftrightarrow \left\langle C_{eq} \middle| \frac{k}{4} \right\rangle, \quad (3.9)$$

showing that the action of Ω on open string fields with boundary condition $|k/4\rangle$ for the orientifold $|C_{eq}\rangle$ is trivial.

Similarly, the Möbius strip for the crosscap $|C_{\pm}\rangle$ is:

$$M_{k/4} = (-1)^{k/2} \sum_{j=0}^{k/2} (-1)^j \chi_j \longleftrightarrow \left\langle C_{\pm} \middle| \frac{k}{4} \right\rangle, \quad (3.10)$$

showing that the projection (3.6) represents the action of Ω on open string fields. More precisely, (3.6) is induced by the crosscap $|C_{\pm}\rangle$ in the case that $k = 0 \bmod 4$ and by $-|C_{\pm}\rangle$ in the case that $k = 2 \bmod 4$.

In the $SU(2)$ case, the involution $g \rightarrow g^{-1}$ actually leaves all conjugacy classes set-wise fixed. One easily sees that the action on the representatives g_J written down in (3.2) is conjugation by the element

$$k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.11)$$

All other elements in the conjugacy class hg_Jh^{-1} can then be inverted by conjugation with hkh^{-1} . In this way, the inversion $g \rightarrow g^{-1}$ can be understood as an inversion within the conjugacy classes. This is precisely what the Möbius strip reflects:

$$\sum_{j=0}^{2J} (-1)^j \chi_j \longleftrightarrow (-1)^{2J} \langle C_{\pm} | J \rangle = (-1)^{2(k/2-J)} \left\langle C_{\pm} \middle| \frac{k}{2} - J \right\rangle, \quad (3.12)$$

where $J < k/4$. The representations appearing in the amplitude are the same ones as for the cylinder amplitude with boundary conditions J or $k/2 - J$. The reflection is encoded in the non-trivial operation of Ω on those characters. Note that the results for the Möbius strip for the brane labeled $|J\rangle$ and $|k/2 - J\rangle$ agree. This is easily understood in geometrical terms since these pairs of branes have symmetric locations on the upper and lower hemisphere of S^2 . In particular, their location is symmetric with respect to the two orientifold fixed point sets at the poles.

For the other involution $g \rightarrow -g^{-1}$, all conjugacy classes except the one located on the equator, get mapped to other conjugacy classes. On the level of representatives, we see that

$$g_J \rightarrow g_{k/2-J}. \quad (3.13)$$

Branes are mapped to image branes, and the Möbius amplitudes contain states propagating between a brane and its image brane. In the language of Cardy states, the image branes are obtained by fusion of the Cardy label with the simple current, mapping $|J\rangle$ to $|k/2 - J\rangle$.

There are two predictions for the Möbius amplitudes arising from these observations. The first one is that the Möbius amplitude with boundary condition $|J\rangle$ agrees with the one with the image boundary condition $|k/2 - J\rangle$. The second one is that the characters appearing in the loop channel of the Möbius strip with boundary condition $|J\rangle$ should be the ones appearing in the cylinder with boundary conditions $|J\rangle$ and $|k/2 - J\rangle$.

Indeed, an explicit computation of the Möbius amplitudes yields the following result:

$$M^J = M^{k/2-J} = \sum_{l=0}^{k/2} \frac{1}{\sqrt{k+2}} (-1)^l \frac{\sin\left(\pi \frac{(2l+1)(2J+1)}{k+2}\right)}{\sin\left(\pi \frac{(2l+1)}{2(k+2)}\right)} \chi_l \longleftrightarrow \sum_{j=\frac{k}{2}-2J}^{k/2} \chi_j, \quad (3.14)$$

confirming the geometrically motivated predictions.

This concludes the interpretation of all Möbius strips in the $SU(2)_k$ model in terms of geometric data.

4. The shape computation

The shape of the two orientifold planes can be determined by scattering massless closed string states. Geometrical results are only expected in the limit $k \rightarrow \infty$. This has been done for boundary states in [5, 26, 4]. Since we will closely follow the discussion in [5, 26], let us briefly review their argument. Scattering amplitude between boundary- or crosscap states and massless closed strings are computed as overlaps of the boundary (crosscap) states with closed string ground states, which are in the case of $SU(2)$ the states $|j, m, m'\rangle$, where no descendant appears. To determine the location of the orientifold, it is useful to pick a graviton wave packet localized at a point of the group manifold. To write down such a δ -function shaped closed string state, [5, 26] made use of the Peter-Weyl theorem which states that the space of functions $\mathcal{F}(G)$ on a group manifold is isomorphic to an infinite direct sum of tensor products of irreducible representations. The matrix elements of the irreducible representations form a complete orthogonal basis for the space of functions $\mathcal{F}(G)$. For $SU(2)$ the following rescaled basis is an orthonormal basis with respect to the Haar measure on the group:

$$\sqrt{2j+1} D_{mm'}^j(g) = \langle jm | R(g) | jm' \rangle. \quad (4.1)$$

In this basis a closed string δ -function on the group manifold reads:

$$|g\rangle = \sum_j \sqrt{2j+1} |j, m, m'\rangle. \quad (4.2)$$

However, one also wants that the closed string probe contains only states with low j , $j^2 \leq k$, since only in that case the closed string states are well localized. To suppress higher j one can introduce an explicit cutoff factor $\exp(-j^2/k)$ into the above sum, which suppresses string modes with $j \geq \sqrt{k}$.

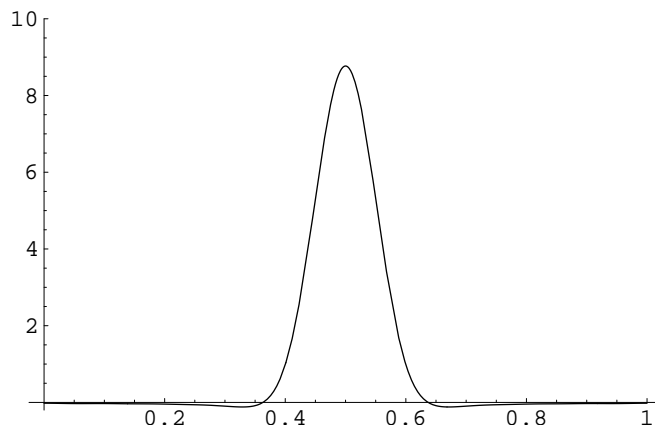


Figure 1: The shape function $s(\psi)$ for $|C_{eq}\rangle$. ψ is plotted in multiples of π .

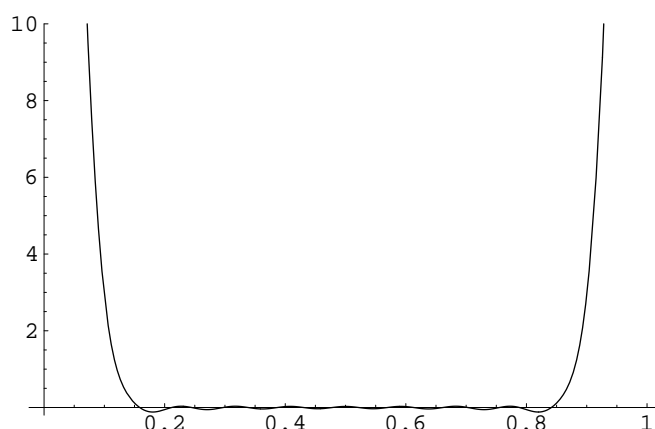


Figure 2: The shape function for $|C_{\pm}\rangle$.

In [5] the $SU(2)$ group manifold is parametrized by three angles, summarized in a vector $\vec{\theta}$. The only angle which we need explicitly is the angle ψ , which labels the $SU(2)$ conjugacy classes. If we want to relate this to our earlier discussion at finite k , we can set $\psi = 2\pi iJ/k$, in particular, $\psi = 0, \dots, \pi$, and 1 and -1 are located at $\psi = 0$ and $\psi = \pi$. The other two angle variables parametrize the spheres $S^2 \subset S^3$, and will not enter the discussion explicitly. A group element in the conjugacy class labeled ψ will be denoted g_ψ , in analogy to our earlier notation g_J . [5, 26] then proceed and take overlaps of boundary states with δ -function states (4.2). To evaluate the expressions, one needs the relation of the matrix elements $D_{mm'}^j$ to the $SU(2)$ characters:

$$\sum_m D_{mm}^j(g_\psi) = \frac{\sin(2j+1)\psi}{\sin \psi}. \quad (4.3)$$

We now want to do an analogous computation for the crosscap states $|C_{eq}\rangle$ and $|C_{\pm}\rangle$. The difference between the boundary state and the crosscap computation is that the boundary state is a linear combinations of conventional Ishibashi states, whereas the crosscap is

a combination of the crosscap Ishibashi states (2.4). However, since the closed string states with which we take the overlap are ground states, this does not modify the computation.

Another important point to keep in mind is that the coefficients of the crosscap Ishibashi states can only be non-vanishing when the primary on which the Ishibashi state is built is invariant under Ω . Therefore, the summation is only over a subset of representations and the crosscap is less well localized compared to boundary states.

In the case at hand only primaries with even j enter the crosscap state. Ω -invariant δ functions (for which the summation in (4.2) is only over even j) can resolve conjugacy classes only up to reflections around the equator, meaning, they cannot distinguish the location at g_J from the location at $g_{k/2-J}$. The consequence is of course that the orientifold loci have to be symmetric around the equator. D-brane boundary states can only be localized up to reflections around the equator by invariant δ -functions. This is another point of view one can take when interpreting the Möbius amplitudes computed earlier.

Keeping all this in mind, we can perform the analogous computation for crosscaps:

$$\begin{aligned}\langle C_{eq}|g_\psi\rangle &\sim \sum_{j=0}^{k/2} \sum_m (-1)^j \cot^{1/2} \left(\pi \frac{2j+1}{2(k+2)} \right) \sqrt{2j+1} D_{mm'}^j(g_\psi)^* \\ &\sim \sum_j (-1)^j \frac{\sin(2j+1)\psi}{\sin \psi} \\ &\sim \delta \left(\psi - \frac{\pi}{2} \right).\end{aligned}\tag{4.4}$$

The approximation made passing from the first to the second line is only good if $j^2 \ll k$. One can make this more explicit in the equations by using the explicit cutoff $\exp(-j^2/k)$. In this way we obtain the “shape-function” $s(\psi)$:

$$s(\psi) = \left(\frac{2}{k+2} \right)^{1/4} \sum_{j=0}^{k/2} e^{-\frac{j^2}{k}} (-1)^j \cot^{1/2} \left(\frac{(2j+1)\pi}{2(k+2)} \right) \sqrt{2j+1} \frac{\sin(2j+1)\psi}{\sin \psi}.\tag{4.5}$$

This function has been plotted for $k = 20$ in picture 1.

This clearly shows that the orientifold locus is at $\psi = \pi/2$.

Let us move on to the second crosscap. We first make the same approximations as before:

$$\begin{aligned}\langle C_\pm|g_\psi\rangle &\sim \sum_{j=0}^{k/2} \sum_m \tan^{1/2} \left(\pi \frac{2j+1}{2(k+2)} \right) \sqrt{2j+1} D_{mm'}^j(g_\psi)^* \\ &\sim \sum_j \sin \left(\frac{\pi(2j+1)}{k+2} \right) \frac{\sin(2j+1)\psi}{\sin \psi} \\ &\sim \delta(\psi - \pi) + \delta(\psi).\end{aligned}\tag{4.6}$$

Again, the approximation leading to the second line is bad unless big j are explicitly suppressed. The shape function for this case is given by:

$$s(\psi) = \left(\frac{2}{k+2} \right)^{1/4} \sum_{j=0}^{k/2} e^{-\frac{j^2}{k}} \tan^{1/2} \left(\frac{(2j+1)\pi}{2(k+2)} \right) \sqrt{2j+1} \frac{\sin(2j+1)\psi}{\sin \psi}.\tag{4.7}$$

It is plotted in picture 2.

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